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On a One-Dimensional Model for the Three-Dimensional Vorticity Equation

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The one-dimensional model for the three-dimensional vorticity equation proposed by Constantin, Lax, and Majda is discussed. Some unsatisfactory points are examined, especially when the viscosity is introduced. A different model is suggested, which, while less solvable than the previous one, can be more strictly connected with the three-dimensional vorticity behavior. The study is of interest for the numerical treatment of the three-dimensional vorticity equation.

KEY WORDS: Three-dimensional vorticity equation; Biot-Savart formula; Hilbert transform; one-dimensional model; breakdown of solutions; scale invariance; Lagrangian coordinates.

1. INTRODUCTION

Recently Constantin *et al.*[2] proposed a simple one-dimensional model for the three-dimensional vorticity equation for an inviscid incompressible fluid. The interest of the model is that, while it is simple, by its explicit solution many of the qualitative features that one expects for the three-dimensional vorticity equation are recovered.

Subsequently Schochet[5] added the viscosity to the model and obtained results, again exhibiting explicit solutions, that are in some way strange.

Having in mind that a one-dimensional approximate model, if it is good, can be of great help in the difficult problem of the numerical integration of the three-dimensional vorticity equation, it seems advisable to reconsider the previous models in order to discuss the points that are not completely satisfactory and try to modify them in such a way that the agreement with the three-dimensional vorticity equation is more evident.

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To understand better the models and the modifications suggested, it is useful to state very briefly the principal properties of the three-dimensional vorticity equation and of the model of Constantin *et al.* in order to see why this is so simple, but at the same time to clarify the observations that force us to change the model and lose part of its simplicity.

2. VORTICITY EQUATION

The Euler equation

$$\frac{\partial}{\partial t} v + (v\nabla) v = -\nabla p + f \tag{1}$$

with a forcing f coming from a gradient field can be written as a vorticity equation

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (v\nabla)\omega = (\omega\nabla) v \tag{2}$$

where

$$\omega = \operatorname{rot} v \tag{3}$$

and the incompressibility condition is expressed by

$$\operatorname{div} v = 0 \tag{4}$$

Equation (2) has to be solved by substituting in it the solution v of the differential problem (3), (4). We suppose that the domain of the equation is the whole R^3 and that $\omega(\pm\infty) = 0$.

As is well known,[1] the solution of the problem (3), (4) is given by the Biot-Savart formula

$$\begin{aligned} v(x, t) &= \frac{1}{4\pi} \int \nabla \frac{1}{|x - x'|} \times \omega(x', t)\, dx' \\ &= -\frac{1}{4\pi} \int \frac{x - x'}{|x - x'|^3} \times \omega(x', t)\, dx', \qquad x, x' \in R^3 \end{aligned} \tag{5}$$

With this expression for v it is easy to see that the second member of (2) becomes singular and is defined only as a Cauchy principal value. Considering the matrix ∇v decomposed in its symmetric part

$$D(v) = \tfrac{1}{2}[\nabla v + (\nabla v)^T] \tag{6}$$

and its antisymmetric part

$$J(v) = \tfrac{1}{2}[\nabla v - (\nabla v)^T] \tag{7}$$

which can in a standard way be identified with the vorticity, it is easy to see that

$$J(v)\omega = 0$$

and so Eq. (2) reduces to

$$\frac{D}{Dt}\omega = D(\omega)\omega, \qquad x \in R^3 \tag{8}$$

[we preserve the notation of Constantin *et al.* and $D(\omega)$ holds for (6) when (5) is substituted in it]. From (5) and (8) it follows, for symmetry reasons, that

$$\int D(\omega)\omega \, dx = 0, \qquad x \in R^3 \tag{9}$$

and so

$$\frac{d}{dt}\int \omega(x, t) \, dx = 0, \qquad x \in R^3 \tag{10}$$

because

$$\int (v\nabla)\omega \, dx = 0$$

Then we have the conservation of the total vorticity. Observe that in R^2 we would have the conservation of the vorticity for any fluid particle

$$\frac{D}{Dt}\omega = 0, \qquad x \in R^2 \tag{11}$$

We suppose initially

$$\int \omega(x, 0) \, dx = 0 \tag{12}$$

and null values at infinity: $\omega(\pm\infty, 0) = 0$.

Just to see if (8)–(12) are compatible with a possible explosion of ω, we consider the equation

$$\frac{D\omega}{Dt}\omega \equiv \frac{\partial}{\partial t}\frac{\omega^2}{2} + (v\nabla)\omega \cdot \omega \tag{13}$$

$$= D(\omega)\,\omega^2, \qquad x \in R^3 \tag{14}$$

where we suppose clear the meaning of the quadratic form in the second member. Integrating (14) and considering that the boundary conditions for ω give

$$\int (v\nabla)\omega \cdot \omega \, dx = 0, \qquad x \in R^3 \tag{15}$$

we have the equation

$$\frac{d}{dt}\int \frac{\omega^2}{2}\, dx = \int \omega \frac{\partial \omega}{\partial t}\, dx = \int D(\omega)\, \omega^2 \, dx \tag{16}$$

so, even if the vorticity stays zero in the mean, it can increase without bounds in absolute value.

3. ONE-DIMENSIONAL MODEL

The ingenious idea of Constantin *et al.* was to write the one-dimensional analogue of Eq. (8). The singular matrix $D(\omega)$ constructed by taking the gradient of (5) has as a unique one-dimensional counterpart the Hilbert transform of ω,

$$H(\omega) = \frac{1}{\pi}\, \mathrm{PV} \int_{-\infty}^{\infty} \frac{\omega(x')}{x - x'}\, dx', \qquad x, x' \in R^1 \tag{17}$$

which shows the same kind of singularity of $D(\omega)$. Then, substituting D/Dt with $\partial/\partial t$, they proposed the equation

$$\frac{\partial \omega}{\partial t} = H(\omega)\omega, \qquad x \in R^1 \tag{18}$$

as a one-dimensional model for the three-dimensional vorticity equation. We will explain now why this equation is in effect so simple. The lucky fact is that $H(\omega)$ can be considered as the boundary value of a harmonic function conjugate to ω. More precisely, we can consider $\omega(x, t)$ harmonically extended in the half-plane $-\infty < x < \infty$, $0 < y$, i.e.,

$$\omega(x, t) = \lim_{y \to 0} u(x, y, t) = \lim_{y \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\omega(x', t)}{(x - x')^2 + y^2}\, dx' \tag{19}$$

$$= \lim_{y \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial y} \ln[(x - x')^2 + y^2]\, \omega(x', t)\, dx' \tag{20}$$

The harmonic function conjugate to u is easily found from (20) to be

$$V(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \ln[(x - x')^2 + y^2]\, \omega(x', t)\, dx' \tag{21}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x - x'}{(x - x')^2 + y^2}\, \omega(x', t)\, dx' \tag{22}$$

so

$$H(\omega) = \lim_{y \to 0} V(x, y, t)$$

The conclusion is that we can consider the function $f(z) = u + iV$, holomorphic in the half-plane $y > 0$, solve Eq. (18) in terms of such a function, and then have

$$\omega = \lim_{y \to 0} \operatorname{Re} f(z, t)$$

Incidentally, we have also

$$H(\omega) = \lim_{y \to 0} \operatorname{Im} f(z, t)$$

Equation (18), extended in the half-plane, is

$$\frac{\partial}{\partial t} u = uV \tag{23}$$

and so it is the real part of

$$\frac{\partial f}{\partial t} = -\frac{i}{2} f^2 \tag{24}$$

where $f(z)$ is the holomorphic function

$$f(z, t) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\omega(x', t)}{z - x'}\, dx', \qquad z = x + iy$$

So Eq. (18) has been transformed into an easily solvable quadratic equation; the solution is

$$\omega(x, t) = \frac{4\omega_0(x)}{[2 - tH(\omega_0(x))]^2 + t^2 \omega_0(x)^2} \tag{25}$$

specified by the initial condition $\omega(x, 0) = \omega_0(x)$. Defining the velocity $v(x, t)$ as the primitive of $\omega(x, t)$,

$$v(x, t) = \int_{-\infty}^{x} \omega(x', t)\, dx' \tag{26}$$

we have that the solution of the model is complete.

Remark. From Eq. (18) and from its explicit solution (25) it is clear that the variable x in fact does not take part in the equation: it is only a parameter that characterizes the initial condition, i.e., for any $x \in R^1$ we have a solution of the quadratic ordinary differential equation (24), with the initial condition specified by $\omega_0(x)$. Everything in x depends only on the value of the field in x. In other words, a convective contribution is absent from (18) and we have no interaction between the neighboring vortices, apart from the initial condition of regularity of $\omega_0(x)$.

From the explicit solution (25) it follows that we can have a breakdown of smooth solutions of the one-dimensional vorticity equation (18) if and only if there exists a point x_0 where $\omega_0(x_0) = 0$ and $H(\omega_0)(x_0) > 0$, and (25) also gives the value of the explosion time T.

The interesting result of Constantin *et al.* is that, if these points x_0 are simple zeros of $\omega_0(x_0)$, then

$$\text{(i)} \qquad \lim_{t \to T} \int_{-\infty}^{\infty} |\omega(x, t)|^p\, dx = \infty, \qquad 1 \leqslant p < \infty \tag{27}$$

$$\text{(ii)} \qquad \int_{-\infty}^{\infty} |v(x, t)|^p\, dx \leqslant c_p, \qquad 1 \leqslant p < \infty \quad \text{any } t < T \tag{28}$$

So this model has the following interesting properties similar to the three-dimensional vorticity equation:

1. It has the scale invariance property, like the true vorticity equation (8) (see later).

2. It shows the finite explosion time that one expects in three dimensions [the condition $H(\omega_0)(x_0) > 0$ could be understood as the analogue in three dimensions of $D(\omega)$ expanding in the direction of ω].

3. The energy is not preserved as in three dimensions, but at least it is bounded [(28) with $p = 2$].

To complete the model, Schochet[5] considered Eq. (18) and added the viscosity

$$\frac{\partial \omega}{\partial t} = H(\omega)\omega + \nu\omega_{xx} \tag{29}$$

and equally gave explicit solutions.

The unsatisfactory properties of this model are:

1. Equation (29) has a nonzero stationary solution with null boundary values.
2. The energy of the solution found by Schochet is unbounded.
3. Its explosion time T can be shorter than the case $v = 0$.

4. DIFFICULTIES OF THE MODEL

This situation induces us to discuss the model and try to understand why it does not work completely well and how it can be modified in order to restore some closer similarity with the three-dimensional flow.

1. First of all, in (2) we have two terms from $\mathrm{rot}[(v \cdot \nabla)v]$: $(v \cdot \nabla)\omega$ and $(\omega \cdot \nabla)v$. In Eq. (18) only the second is considered: i.e., knowing that we are in R^1, D/Dt has been substituted with $\partial/\partial t$, as it would be for an incompressible fluid. Now, hoping to construct a one-dimensional representative for the three-dimensional vorticity, we would be ready to expect that the model will not be a one-dimensional incompressible fluid. In other words, of the two differential operators ∇v and $\nabla \times v$, only one survives in R^1 and it seems more convenient to connect, as was done in the paper of Constantin *et al.*, $\partial v/\partial x$ with ω. But if things are done in this way, we have to preserve also the term $(v \cdot \partial/\partial x)\omega$. A further reason for this is that the two terms $(v \cdot \partial/\partial x)\omega$ and $(\omega \cdot \partial/\partial x)v$ are both zero in mean and have the same importance. To see this, observe that

$$\int_{-\infty}^{\infty} \left(v \frac{\partial}{\partial x}\right) \frac{\omega^2}{2}\, dx = v \frac{\omega^2}{2}\bigg|_{-\infty}^{\infty} - \frac{1}{2}\int_{-\infty}^{\infty} \omega^2 v_x \, dx$$
$$= -\frac{1}{2}\int_{-\infty}^{\infty} \omega^2 v_x \, dx$$

So it is more correct to preserve both of them in the one-dimensional model.

2. The second point to consider is that, having written the second member of (18) as $H(\omega)\omega$ instead of ω^2, it was explicitly supposed that $(\partial/\partial x)v \neq \omega$, so we cannot define now

$$v(x, t) = \int_{-\infty}^{x} \omega(x', t)\, dx' \tag{30}$$

A further reason for not adopting the definition (30) is that the solution (5) of the differential problem (3), (4) depends, as expected, on the values of $\omega(x, t)$ on the whole space, while in (30) the values of $\omega(x', t)$ for $x' > x$ do not affect the value of v in x.

3. The third reason to make some changes is that it has been already proved[3,7] also in R^3 that, for some ν_0 and for $\nu \leqslant \nu_0$, the solutions v_ν of the Navier–Stokes equation exist for a time T_{ν_0} and tend uniformly to v_0, with v_0 the solution of the Euler equation. So the solution found by Schochet, with the explosion of the energy for any $\nu > 0$, is not compatible with this theorem.

5. MODEL PROPOSED

For all these reasons we suggest changing the definition of v and not substituting D/Dt with $\partial/\partial t$. As regards the first problem, if the Hilbert transform $H(\omega)$ stays for $\partial v/\partial x$, it is natural to define

$$v(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \ln |x - x'| \, \omega(x', t) \, dx' \tag{31}$$

or, equivalently, if $\omega(\pm\infty, t) = 0$, as we suppose,

$$v(x, t) = \frac{1}{\pi} \operatorname{PV} \int_{-\infty}^{\infty} \frac{\Omega(x', t)}{x - x'} \, dx' = H(\Omega)(x, t) \tag{32}$$

where

$$\Omega(x, t) = \int_{-\infty}^{x} \omega(x', t) \, dx'$$

This means that the present definition of v is in fact the Hilbert transform of the velocity as defined by Constantin *et al.*

So $v(x, t)$ is well defined if $\omega(x, t)$ is integrable, decaying sufficiently well at infinity, and depends, as desired, on all the values of ω. Further, since $\Omega(\pm\infty, t) = 0$, it follows that also $v(\pm\infty, t) = 0$. [One has

$$v(x, t) = \frac{1}{\pi} \ln |x| \int_{-\infty}^{\infty} \omega(x', t) \, dx' + \frac{1}{\pi} \int_{-\infty}^{\infty} \ln \frac{|x - x'|}{|x|} \, \omega(x', t) \, dx'$$

and $\Omega(\pm\infty, t) = 0$ is a necessary and sufficient condition to have $v(\pm\infty, t) = 0$.]

The one-dimensional model for the three-dimensional vorticity equation we suggest is then

$$\frac{D\omega}{Dt} \equiv \frac{\partial \omega}{\partial t} + v\omega_x = \omega v_x \equiv \omega H(\omega) \tag{33}$$

where v is defined by (31) or (32).

Let us now examine the properties of this model.

1. First of all, we cannot now use the reasoning used to solve (18): Eq. (33) is no longer the boundary value of the real part of a holomorphic equation; even if ω and v_x (u and V in the previous notations) are harmonic, the product $v\omega_x$ is not harmonic.

2. The solution of (33) still satisfies the conservation law

$$\frac{d}{dt}\int_{-\infty}^{\infty} \omega(x, t)\, dx = 0$$

while

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}\omega^2(x, t)\, dx = \frac{3}{2}\int_{-\infty}^{\infty} \omega^2 H(\omega)\, dx \tag{34}$$

Observe that in the first member of (34) there is a contribution from the compressibility of the fluid. More precisely, while in R^2 and R^3

$$\int (v \cdot \nabla)\omega \cdot \omega\, dx$$

is zero for the div $v = 0$ condition, now we have

$$\int_{-\infty}^{\infty} (v \cdot \partial x)\frac{1}{2}\omega^2\, dx = -\frac{1}{2}\int_{-\infty}^{\infty} \omega^2 H(\omega)\, dx$$

Considering a fixed interval (x_1, x_2), we have

$$\frac{d}{dt}\int_{x_1}^{x_2} \frac{1}{2}\omega^2\, dx = \frac{3}{2}\int_{x_1}^{x_2} \omega^2 H(\omega)\, dx$$

and so we have that an increase or decrease of ω is due also to a convective contribution [see Remark following Eq. (26)].

3. Equation (33) with (32) satisfies the scale invariance property: if $\omega(x, t)$ is a solution of (33), then

$$\omega_\lambda(x, t) = \lambda^{1+\alpha}\omega(\lambda^\alpha x, \lambda^{1+\alpha} t)$$

satisfies the same equation with

$$v_\lambda(x, t) = \lambda v(\lambda^\alpha x, \lambda^{1+\alpha} t)$$

4. Introducing as an unknown function the inverse of ω,

$$\tau = 1/\omega$$

we see that Eq. (33) assumes a very simple and interesting form,

$$\frac{\partial \tau}{\partial t}+\frac{\partial}{\partial x}(v\tau)=0 \tag{35}$$

i.e., it is a conservation law for τ in a compressible fluid. The decrease of τ (and so the increase of ω) in the interval (x_1, x_2) is connected to the quantity of τ escaping from the extremes of the interval (x_1, x_2). The conservation law says, for example, as we will see better in the following properties, that if τ is in mean bounded and different from zero initially, it cannot become zero (and so ω become ∞) if $|\partial x(t)/\partial x(0)| \leqslant M < \infty$, where $x(t)$ is the solution of the differential equation

$$\dot{x}=v(x, t) \tag{36}$$

with v defined by (32).

Obviously, (36) cannot be solved without having solved (32), (33).

Considering so the Lagrangian form of the equations, and denoting by $\phi_t x$ the solution of (36) that at $t=0$ starts from x, i.e.,

$$\phi_t x=x(t, x, 0) \qquad \text{where} \quad x(0, x, 0)=x$$

we have the following property.

5. In Lagrangian coordinates the formal solution of Eq. (33) looks like the formal solution of the vorticity equation in R^3. One has (see, for example, ref. 6)

$$\frac{\omega(\phi_t x, t)}{\rho(\phi_t x, t)}=\frac{\omega_0(x)}{\rho_0(x)}\frac{\partial \phi_t x}{\partial x}, \qquad x \in R^3 \tag{37}$$

and, for $\rho=\text{const}$,

$$\omega(\phi_t x, t)=\omega_0(x)\frac{\partial \phi_t x}{\partial x}, \qquad x \in R^3 \tag{38}$$

Now we have

$$\omega(\phi_t x, t)=\omega_0(x)\frac{\partial \phi_t x}{\partial x} \tag{39}$$

but (39) is valid for a one-dimensional compressible fluid: the incompressibility condition would give

$$\omega(\phi_t x, t)=\omega_0(x) \tag{40}$$

Remark. The stretching in R^3 (for $\rho = \text{const}$) is connected to the eigenvalues of $\partial\phi_t x/\partial x$; the stretching in R^1 instead is connected to the condition $\rho \neq \text{const}$. In R^3 we can have the Jacobian determinant equal to 1 and nevertheless eigenvalues of the matrix $\partial\phi_t x/\partial x$ greater than 1 in some direction (and less than 1 in other directions); in R^1, on the contrary, the matrix and the determinant coincide, so we cannot simulate in R^1 at the same time the behavior of the three-dimensional vorticity in all directions.

We also give in explicit form a property already contained in the preceding

6. The explosion of ω is possible if and only if $\partial\phi_t x/\partial x$ is unbounded. And noting that $\partial\phi_t x/\partial x$ is unbounded if and only if $\partial v/\partial x$ is unbounded, we recover a result true for the three-dimensional flow.[4]

We note also that

$$\frac{d}{dt}\int v^2\,dx = \frac{d}{dt}\int \Omega^2\,dx \tag{41}$$

This is obtained by substituting (32) in the left member of (41) and recalling that

$$\int \frac{dx}{(x-x')(x-x'')} = \pi^2\delta(x'-x'') \tag{42}$$

when we consider functions for which the Hilbert transform is defined.

Let us now comment some more on (25) and on the periodic example considered by Constantin *et al.* If $\omega_0(x) = \cos x$, then $H(\omega_0)(x) = \sin x$ and (25) becomes

$$\omega(x, 2t) = \frac{\cos x}{1 + t^2 - 2t\sin x} \tag{43}$$

So the point $x = \pi/2$ satisfies the request that there be an explosion for ω: the explosion time T is now $T = 2$. But in $x = \pi/2$ the solution is always zero: it is in the neighborhood of this point that the solution develops a nonintegrable singularity. This example can then appear artificial and connected in some way with the pathology of trigonometric series. This behavior seems analogous to that of the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}$$

in $x = 0$: the series converges in any point, but around $x = 0$ the sum of the series goes like $1/x$.

In contrast with (43), the velocity v has in $x=\pi/2$ an integrable singularity.

If we look now for a stationary solution of Eq. (33), we will have a slightly surprising result. The stationary solution satisfies the equations

$$\frac{\partial}{\partial x}\frac{v}{\omega}=0, \qquad \frac{v}{\omega}=\text{const} \tag{44}$$

i.e.,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Omega(x')}{x-x'}\,dx'=c\omega(x) \tag{45}$$

or

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\omega(x')}{x-x'}\,dx'=c\omega'(x) \tag{46}$$

There is no problem of additive constants, for we have $\omega(\pm\infty)=\Omega(\pm\infty)=0$. The unique solutions of these equations are, as is easy to verify, the trigonometric functions. In this case the condition

$$\int_{-\infty}^{\infty}\omega\,dx=0$$

reads

$$\int_{-\pi}^{\pi}\omega\,dx=0$$

So the exploding solution found by Constantin *et al.* becomes a stationary solution of the present model (the convective term forbids the explosion of ω).

The difficulties posed by the work of Schochet also can now be solved very easily. He started from a stationary solution of (29), and from that constructed a nonstationary solution. But that stationary solution does not exist for $\nu=0$, and so it is not surprising that the solution he found for Eq. (29) does not satisfy the theorem of uniform convergence to the solutions of (18), as we would expect if the model were a good model for the three-dimensional vorticity.

If we consider the Navier–Stokes analogue of our model, we have

$$\frac{\partial\omega}{\partial t}+v\omega_x=H(\omega)\omega+\nu\omega_{xx} \tag{47}$$

and this equation, as we would expect, has no stationary solution different from zero for $v > 0$ and $\omega(\pm\infty) = 0$. So the previous difficulty has been overcome and this is a further support for the present model.

Further properties of the model and applications to the numerical integration of the three-dimensional vorticity equation will be considered in a further paper.

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